Question 1) [10 marks]

a) Determine the equivalent transfer function for the following state space equation.

[3 marks]

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

b) Write down an equivalent state space equation for the following transfer function.

[3 marks]

$$G(s) = 7\frac{(s+1)}{(s+8)(s+5)}$$

c) Identify the stability of a dynamic system governed by the following state space equation.

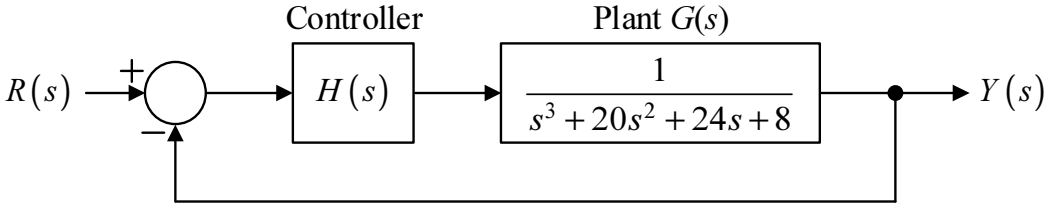
[4 marks]

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t)$$

Question 2) [10 marks]

A PID controller is used to regulate the output of a linear dynamic system as shown in the feedback block diagram below. The gains of the PID controller are set using a Ziegler-Nichols tuning strategy.



a) Find the critical gain, and the associated period of oscillation, at the point where the closed loop system operating with proportional action only achieves sustained un-damped oscillation.

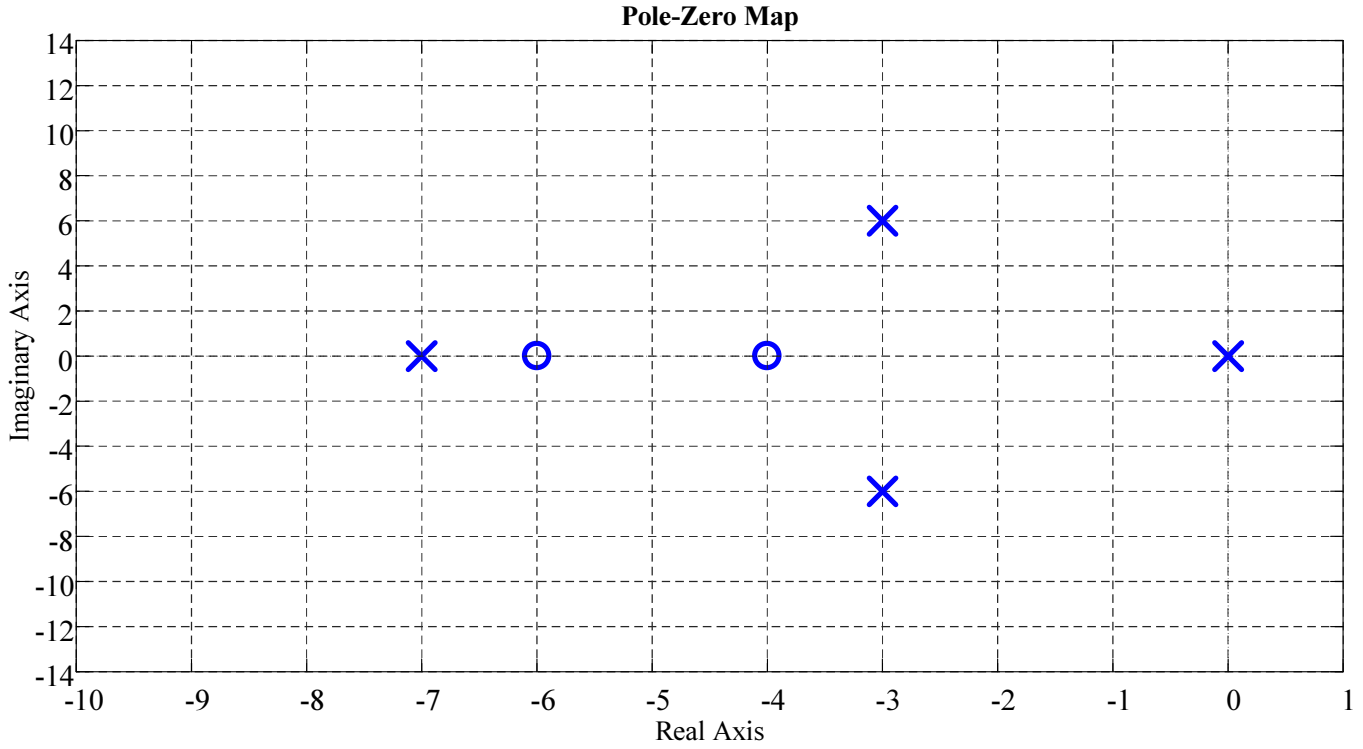
[5 marks]

b) Using your result from a), determine the gain parameters required for the PID controller, and hence write down an expression for the PID controller transfer function.

[5 marks]

Question 3) [10 marks]

Consider the pole-zero map illustrated below, which provides the locations of all open loop poles and zeros present in the forward path gain transfer function $GH(s)$ of a standard negative feedback control system. You are required to sketch the root locus of closed loop poles for this system.



a) Plot on the pole-zero map any real axis branches of the root locus, if they exist.

[2 marks]

b) Determine the asymptote features (i.e. centroid location and asymptote angles) and then plot the asymptotes on the pole-zero map.

[3 marks]

c) Calculate the departure and/or arrival angles for any complex-valued poles or zeros.

[3 marks]

d) Hence or otherwise, sketch the complete root-locus plot for this system on the pole-zero map shown.

[2 marks]

End of Test

Examination Formula Sheet

Laplace Transforms

$f(t),\ t\geq 0$	$\mathcal{L}\{f(t)\}$	$f(t),\ t\geq 0$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}F(s)\,ds$	$\int_0^{\infty} e^{-st}f(t)\,dt$
$\delta(t)$	1	$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$
t	$\frac{1}{s^2}$	$\frac{dy(t)}{dt}$	$sY(s)-y(0)$
$t^n,\ \ n\in\{2,3,4,\ldots\}$	$\frac{n!}{s^{n+1}}$	$\frac{d^k y(t)}{dt^k}$	$s^kY(s)-\sum_{i=1}^{k} s^{k-i}\frac{d^{i-1}y(0)}{dt^{i-1}}$
e^{at}	$\frac{1}{s-a}$	$\int_0^t y(\tau)\,d\tau$	$\frac{1}{s}Y(s)$
te^{at}	$\frac{1}{(s-a)^2}$	$y(t-\tau)\mu(t-\tau)$	$e^{-s\tau}Y(s)$
$cos(\omega_o t)$	$\frac{s}{s^2+\omega_o{}^2}$	$ty(t)$	$-\frac{dY(s)}{ds}$
$sin(\omega_o t)$	$\frac{\omega_o}{s^2+\omega_o{}^2}$	$t^k y(t)$	$(-1)^k\frac{d^kY(s)}{ds^k}$
$e^{at}\,cos(\omega_o t)$	$\frac{s-a}{(s-a)^2+\omega_o{}^2}$	$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$	$F_1(s)F_2(s)$
$e^{at}\,sin(\omega_o t)$	$\frac{\omega_o}{(s-a)^2+\omega_o{}^2}$	$\lim_{t\to\infty} y(t)$	$\lim_{s\to 0} sY(s)$
$t\,cos(\omega_o t)$	$\frac{s^2-\omega_o{}^2}{(s^2+\omega_o{}^2)^2}$	$\lim_{t\to 0} y(t)$	$\lim_{s\to\infty} sY(s)$
$t\,sin(\omega_o t)$	$\frac{2\omega_o s}{(s^2+\omega_o{}^2)^2}$	$f_1(t)f_2(t)$	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F_1(\xi)F_2(s-\xi)\,d\xi$
$\mu(t)-\mu(t-\tau)$	$\frac{1-e^{-s\tau}}{s}$	$e^{at}f_1(t)$	$F_1(s-a)$

Transfer Functions

$$G(s)=\frac{a_n s^{n-1}+a_{n-1}s^{n-2}+\cdots+a_2 s+a_1}{s^n+b_n s^{n-1}+b_{n-1}s^{n-2}+\cdots+b_2 s+b_1}$$

$$G(s)=K\frac{\prod_{l=1}^{m}(s+z_l)}{\prod_{k=1}^{n}(s+p_k)} \qquad G(s)=\sum_{k=1}^{n}\frac{c_k}{s+p_k} \qquad c_k=(s+p_k)G(s)\Big|_{s=-p_k}$$

$$G(s)=K\frac{\prod_{l=1}^{m}(s+z_l)}{\prod_{k=1}^{r}(s+p_k)^{n_k}} \quad G(s)=\sum_{k=1}^{r}\sum_{l=1}^{n_i}\frac{c_{ik}}{(s+p_k)^l} \quad c_{ik}=\frac{1}{(n_k-l)!}\frac{d^{n_{k-1}}}{ds^{n_{k-1}}}\left[(s+p_k)^{n_k}G(s)\right]\Big|_{s=-p_k}$$

Second Order Canonical System

$$G(s)=\frac{\omega_o^{\,2}}{s^2+2\zeta\omega_o s+\omega_o^{\,2}} \qquad y(t)=1-\frac{e^{-\zeta\omega_o t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\beta)$$

$$\omega_d=\omega_o\sqrt{1-\zeta^2} \qquad \beta=\sin^{-1}\left(\sqrt{1-\zeta^2}\right) \qquad t_r=\frac{\pi-\beta}{\omega_d} \qquad M_p=e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\zeta=\sqrt{\frac{\left[ln(M_p)\right]^2}{\pi^2+\left[ln(M_p)\right]^2}} \qquad e^{-\zeta\omega_o t_s}=\delta \qquad \zeta\omega_n=\frac{-ln(\delta)}{t_s}$$

Frequency Response

$$G(s)=K\frac{\prod_{l=1}^{m}(s/z_l+1)}{s^k\prod_{h=1}^{n}(s/p_h+1)}$$

$$\left|G(j\omega)\right|_{dB}=20log_{10}|K|-20k\,log_{10}|\omega|+\sum_{l=1}^{m}20log_{10}|j\omega/z_l+1|-\sum_{h=1}^{n}20log_{10}|j\omega/p_h+1|$$

$$\angle G(j\omega)=\angle(K)-k\frac{\pi}{2}+\sum_{l=1}^{m}\angle[j\omega/z_l+1]-\sum_{h=1}^{n}\angle[j\omega/p_h+1]$$

State Space Equations

$$\begin{aligned}\frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}\,\mathbf{x}(t)+\mathbf{B}\,\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t)+\mathbf{D}\,\mathbf{u}(t)\end{aligned} \qquad G(s)=\frac{Y(s)}{U(s)}=\mathbf{C}[s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}+\mathbf{D} \qquad \lambda_i,\ \ det[\lambda\mathbf{I}-\mathbf{A}]=0 \qquad \mathbf{A}v_i=\lambda_i v_i$$

$$\frac{d\mathbf{x}(t)}{dt}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 1\\ -b_1 & -b_2 & \cdots & -b_{n-1} & -b_n\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}a_1 & a_2 & \cdots & a_{n-1} & a_n\end{bmatrix}\mathbf{x}(t)+[0]\,u(t)$$

$$\frac{d\mathbf{x}(t)}{dt}=\begin{bmatrix}0 & 0 & \cdots & 0 & -b_1\\ 1 & 0 & \cdots & 0 & -b_2\\ \vdots & 1 & \ddots & \vdots & \vdots\\ 0 & \ddots & \ddots & 0 & -b_{n-1}\\ 0 & 0 & \cdots & 1 & -b_n\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}a_1\\a_2\\\vdots\\a_{n-1}\\a_n\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}\mathbf{x}(t)+[0]\,u(t)$$

$$\frac{d\mathbf{x}(t)}{dt}=\begin{bmatrix}-\lambda_1 & 0 & \cdots & \cdots & 0\\ 0 & -\lambda_2 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & -\lambda_{n-1} & 0\\ 0 & \cdots & \cdots & 0 & -\lambda_n\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_{n-1}\\ a_{n-2}\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 1 & \cdots & 1 & 1\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\end{bmatrix}u(t)$$

$$\begin{aligned}\frac{d\mathbf{x}(t)}{dt} &= \mathbf{f}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) & \qquad \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}\Delta\mathbf{x}(t)+\mathbf{B}\Delta\mathbf{u}(t)+\mathbf{E}\\ \mathbf{y}(t) &= \mathbf{g}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) & \qquad \mathbf{y}(t) &= \mathbf{C}\Delta\mathbf{x}(t)+\mathbf{D}\Delta\mathbf{u}(t)+\mathbf{F}\end{aligned}$$

$$\mathbf{A}=\left.\frac{d\mathbf{f}}{d\mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}},\quad \mathbf{B}=\left.\frac{d\mathbf{f}}{d\mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}},\quad \mathbf{C}=\left.\frac{d\mathbf{g}}{d\mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}},\quad \mathbf{D}=\left.\frac{d\mathbf{g}}{d\mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_Q\\ \mathbf{u}=\mathbf{u}_Q}},\quad \mathbf{E}=\mathbf{f}\left(\mathbf{x}_Q,\mathbf{u}_Q\right),\quad \mathbf{F}=\mathbf{g}\left(\mathbf{x}_Q,\mathbf{u}_Q\right)$$

$$\Delta\mathbf{x}(t)=\mathbf{x}(t)-\mathbf{x}_Q,\qquad \Delta\mathbf{u}(t)=\mathbf{u}(t)-\mathbf{u}_Q$$

PID Controllers

	K_P	T_r	T_D
P	$0.5K_C$		
PI	$0.45K_C$	$\frac{P_C}{1.2}$	
PID	$0.6K_C$	$\frac{P_C}{2}$	$\frac{P_C}{8}$

$$H_P(s)=K_P \qquad H_{PI}(s)=K_P+\frac{K_I}{s} \qquad H_{PI}(s)=K_P\left(1+\frac{1}{T_r s}\right)\ T_r=K_P/K_I$$

$$H_{PD}(s)=K_P\left(1+\frac{T_d s}{\tau_D s+1}\right) \qquad H_{PD}(s)=K_P+K_d s \quad T_d=K_D/K_P$$

$$H_{PID}(s)=K_P\left(1+\frac{1}{T_r s}+\frac{T_d s}{\tau_D s+1}\right) \qquad H_{PID}(s)=K_P+\frac{K_I}{s}+K_d s$$

Root Locus

$$G(s)=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{k=1}^{n}(s+p_k)} \qquad \angle G(s)=180°+360°(l) \qquad \angle G(s)=\sum_{i=1}^{m}\psi_i-\sum_{k=1}^{n}\varphi_k$$

$$\psi_i=tan^{-1}\left(\frac{Im(s+z_i)}{Re(s+z_i)}\right) \qquad \varphi_k=tan^{-1}\left(\frac{Im(s+p_k)}{Re(s+p_k)}\right) \qquad b(s)\frac{da(s)}{ds}-a(s)\frac{db(s)}{ds}=0$$

$$\varphi=\frac{180°+360°(l-1)}{n-m},\quad l=1,2,\ldots,n-m \qquad \alpha=\frac{\sum p-\sum z}{n-m}$$

$$q\varphi_{dep}=\sum\psi-\sum\varphi-180°-360°l \qquad q\psi_{arr}=\sum\varphi-\sum\psi+180°+360°l$$

$$K_P = \left|\frac{1}{G(s)}\right| = \frac{|s_o - p_n||s_o - p_{n-1}|\cdots|s_o - p_2||s_o - p_1|}{|s_o - z_m||s_o - z_{m-1}|\cdots|s_o - z_2||s_o - z_1|} \quad |s_o - p_n| = \sqrt{Re(s_o - p_n)^2 + Im(s_o - p_n)^2}$$

Dynamic Compensation

$$D_{lag}(s) = \frac{s+z}{s+p},\ p < z \qquad D_{lead}(s) = \frac{s+z}{s+p},\ z < p \qquad \angle G(s_o) + \angle D_{lead}(s_o) = 180°$$

$$E_k = \lim_{s\to 0}\left[s^k K_p D_{lag}(s) G(s)\right] = K_p \frac{z}{p}\lim_{s\to 0}\left[s^k G(s)\right]$$

Nyquist and Bode Design

$$Z = N + P \qquad \varphi_m \approx 100\zeta$$

$$D_{lag}(s) = \frac{s/z+1}{s/p+1}, \quad z > p, \qquad 20log_{10}\left(\frac{p}{z}\right) = D, \qquad z << \omega_c$$

$$D_{lead}(s) = \frac{s/z+1}{s/p+1}, \quad z < p, \quad \theta_{lead} = 90° - 2tan^{-1}\left[\sqrt{\frac{z}{p}}\right], \qquad \omega_c = \sqrt{zp}$$

$$20log10\left|H_{lag}(j\omega_{sc}) H_{lead}(j\omega_{sc}) G(j\omega_{sc})\right| = 0dB$$